

Partially Observed Optimal Control for Mean-Field SDEs *

Maoning Tang, Qingxin Meng [†]

Department of Mathematics, Huzhou University, Zhejiang 313000, China

Abstract

In this paper, we are concerned with a stochastic optimal control problem of mean-field type under partial observation, where the state equation is governed by the controlled nonlinear mean-field stochastic differential equation, moreover the observation noise is allowed to enter into the state equation and the observation coefficients may depend not only on the control process and but also on its probability distribution. Under standard assumptions on the coefficients, by dual analysis and convex variation, we establish the maximum principle for optimal control in a strong sense as well as a weak one, respectively. As an application, a partially observed linear quadratic control problem of mean-field type is studied detailed and the corresponding dual characterization and state feedback presentation of the partially observed optimal control are obtained by the stochastic maximum principles and the classic technique of completing squares.

Keywords: Maximum Principle, Mean-Field Stochastic Differential Equation, Mean-Field Backward Stochastic Differential Equation, Partial Observation, Girsanov's Theorem

1 Introduction

1.1 Basic Notations

In this subsection, we introduce some basic notations which will be used in this paper. Let $\mathcal{T} := [0, T]$ denote a finite time index, where $0 < T < \infty$. We consider a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ equipped with two one-dimensional standard Brownian motions $\{W(t), t \in \mathcal{T}\}$ and $\{Y(t), t \in \mathcal{T}\}$, respectively. Let $\{\mathcal{F}_t^W\}_{t \in \mathcal{T}}$ and $\{\mathcal{F}_t^Y\}_{t \in \mathcal{T}}$ be \mathbb{P} -completed natural filtration generated by $\{W(t), t \in \mathcal{T}\}$ and $\{Y(t), t \in \mathcal{T}\}$, respectively. Set $\{\mathcal{F}_t\}_{t \in \mathcal{T}} := \{\mathcal{F}_t^W\}_{t \in \mathcal{T}} \vee \{\mathcal{F}_t^Y\}_{t \in \mathcal{T}}$, $\mathcal{F} = \mathcal{F}_T$. Denote by $\mathbb{E}[\cdot]$ the expectation under the probability \mathbb{P} . Let E be a Euclidean space. The inner product in E is denoted by $\langle \cdot, \cdot \rangle$, and the norm in E is denoted by $|\cdot|$. Let A^\top denote the transpose of the matrix or vector A . For a function $\phi : \mathbb{R}^n \rightarrow \mathbb{R}$, denote by ϕ_x its gradient. If $\phi : \mathbb{R}^n \rightarrow \mathbb{R}^k$ (with $k \geq 2$), then $\phi_x = (\frac{\partial \phi_i}{\partial x_j})$ is the corresponding $k \times n$ -Jacobian matrix. By \mathcal{P} we denote the predictable σ field on $\Omega \times [0, T]$ and by $\mathcal{B}(\Lambda)$ the Borel σ -algebra of any topological space Λ . In the follows, K represents a generic constant, which can be different from line to line.

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[†]Corresponding author. E-mail address: mqx@zjhu.edu.cn,

Next we introduce some spaces of random variable and stochastic processes. For any $\alpha, \beta \in [1, \infty)$, we let

- $M_{\mathcal{F}}^{\beta}(0, T; E)$: the space of all E -valued and \mathcal{F}_t -adapted processes $f = \{f(t, \omega), (t, \omega) \in \mathcal{T} \times \Omega\}$ satisfying $\|f\|_{M_{\mathcal{F}}^{\beta}(0, T; E)} \triangleq \left(\mathbb{E} \left[\int_0^T |f(t)|^{\beta} dt \right] \right)^{\frac{1}{\beta}} < \infty$.
- $S_{\mathcal{F}}^{\beta}(0, T; E)$: the space of all E -valued and \mathcal{F}_t -adapted càdlàg processes $f = \{f(t, \omega), (t, \omega) \in \mathcal{T} \times \Omega\}$ satisfying $\|f\|_{S_{\mathcal{F}}^{\beta}(0, T; E)} \triangleq \left(\mathbb{E} \left[\sup_{t \in \mathcal{T}} |f(t)|^{\beta} \right] \right)^{\frac{1}{\beta}} < +\infty$.
- $L^{\beta}(\Omega, \mathcal{F}, P; E)$: the space of all E -valued random variables ξ on (Ω, \mathcal{F}, P) satisfying $\|\xi\|_{L^{\beta}(\Omega, \mathcal{F}, P; E)} \triangleq \sqrt{\mathbb{E}|\xi|^{\beta}} < \infty$.
- $M_{\mathcal{F}}^{\beta}(0, T; L^{\alpha}(0, T; E))$: the space of all $L^{\alpha}(0, T; E)$ -valued and \mathcal{F}_t -adapted processes $f = \{f(t, \omega), (t, \omega) \in [0, T] \times \Omega\}$ satisfying $\|f\|_{\alpha, \beta} \triangleq \left\{ \mathbb{E} \left[\left(\int_0^T |f(t)|^{\alpha} dt \right)^{\frac{\beta}{\alpha}} \right] \right\}^{\frac{1}{\beta}} < \infty$.

1.2 Formulation of Optimal control Problem of Mean-Field Type Under Partial Observation

In this subsection, under partial observations, we formulate two class of optimal control problems of mean-field type in a weak form and a strong form, respectively. On probability space $(\Omega, \mathcal{F}, \mathbb{P})$, we consider the following controlled mean-field stochastic differential equation

$$\begin{cases} dx(t) &= b(t, x(t), \mathbb{E}[x(t)], u(t), \mathbb{E}[u(t)])dt + g(t, x(t), \mathbb{E}[x(t)], u(t), \mathbb{E}[u(t)])dW(t) \\ &\quad + \tilde{g}(t, x(t), \mathbb{E}[x(t)], u(t), \mathbb{E}[u(t)])dW^u(t), \\ x(0) &= a \in \mathbb{R}^n, \end{cases} \quad (1.1)$$

with an obvervation

$$\begin{cases} dY(t) &= h(t, x(t), \mathbb{E}[x(t)], u(t), \mathbb{E}[u(t)])dt + dW^u(t), \\ y(0) &= 0, \end{cases} \quad (1.2)$$

where $b : \mathcal{T} \times \Omega \times \mathbb{R}^n \times \mathbb{R}^n \times U \times U \rightarrow \mathbb{R}^n$, $g : \mathcal{T} \times \Omega \times \mathbb{R}^n \times \mathbb{R}^n \times U \times U \rightarrow \mathbb{R}^n$, $\tilde{g} : \mathcal{T} \times \Omega \times \mathbb{R}^n \times \mathbb{R}^n \times U \times U \rightarrow \mathbb{R}^n$, $h : \mathcal{T} \times \Omega \times \mathbb{R}^n \times \mathbb{R}^n \times U \times U \rightarrow \mathbb{R}$, are given random mapping with U being a nonempty convex subset of \mathbb{R}^k . In the above equations, $u(\cdot)$ is our admissible control process defined as follows.

Definition 1.1. An admissible control process is defined as a stochastic process $u : \mathcal{T} \times \Omega \longrightarrow U$ which is $\{\mathcal{F}_t^Y\}_{t \in \mathcal{T}}$ -adapted and satisfies

$$\mathbb{E} \left[\left(\int_0^T |u(t)|^2 dt \right)^2 \right] < \infty. \quad (1.3)$$

The set of all admissible controls is denoted by U_{ad}^W .

Remark 1.1. In the literature (see, e.g., Tang(1998)), we know that a control process is said to be partially observed if the control is nonanticipative functional of the observation $Y(\cdot)$. A set of controls is said to be partially observed if its element is partially observed. Obviously, the set U_{ad}^W of all admissible control is partially observed.

Now we make the following standard assumptions on the coefficients of the equations (1.1) and (1.2).

Assumption 1.1. The coefficients b, g, \tilde{g} and h are $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}^n) \otimes \mathcal{B}(\mathbb{R}^n) \otimes \mathcal{B}(U) \otimes \mathcal{B}(U)$ -measurable. For each $(x, y, u, v) \in \mathbb{R}^n \times \mathbb{R}^n \times U \times U$, $b(\cdot, x, y, u, v), g(\cdot, x, y, u, v), \tilde{g}(\cdot, x, y, u, v)$ and $h(\cdot, x, y, u, v)$ are all $\{\mathcal{F}_t\}_{t \in \mathcal{T}}$ -adapted processes. For almost all $(t, \omega) \in \mathcal{T} \times \Omega$, the mapping

$$(x, y, u, v) \rightarrow \varphi(t, \omega, x, y, u, v)$$

is continuous differentiable with respect to (x, y, u, v) with appropriate growths, where $\varphi = b, g, \tilde{g}$ and h . More precisely, there exists a constant $C > 0$ such that for all $x, y \in \mathbb{R}^n, u, v \in U$ and a.e. $(t, \omega) \in \mathcal{T} \times \Omega$,

$$\begin{cases} (1 + |x| + |y| + |u| + |v|)^{-1} |\phi(t, x, y, u, v)| + |\phi_x(t, x, y, u, v)| \\ \quad + |\phi_y(t, x, y, u, v)| + |\phi_u(t, x, y, u, v)| + |\phi_v(t, x, y, u, v)| \leq C, \varphi = b, g, \tilde{g}; \\ |h(t, x, y, u, v)| + |h_x(t, x, y, u, v)| + |h_y(t, x, y, u, v)| + |h_u(t, x, y, u, v)| + |h_v(t, x, y, u, v)| \leq C. \end{cases}$$

Now under Assumption 1.1, we begin to discuss the well-posedness of (1.1) and (1.2). Indeed, putting (1.2) into the state equation (1.1), we get that

$$\begin{cases} dx(t) &= [(b - \tilde{g}h)(t, x(t), \mathbb{E}[x(t)], u(t), \mathbb{E}[u(t)])]dt + g(t, x(t), \mathbb{E}[x(t)], u(t), \mathbb{E}[u(t)])dW(t) \\ &\quad + \tilde{g}(t, x(t), \mathbb{E}[x(t)], u(t), \mathbb{E}[u(t)])dY(t), \\ x(0) &= a. \end{cases} \quad (1.4)$$

Under Assumption 1.1, for any $u(\cdot) \in U_{ad}^W$, by Lemma 1.4 below, (1.4) admits a strong solution $x(\cdot) \equiv x^u(\cdot) \in S_{\mathcal{F}}^4(0, T; \mathbb{R}^n)$. On the other hand, for any $u(\cdot) \in U_{ad}^W$ associated with the corresponding solution $x^u(\cdot)$ of (1.4), introduce a stochastic process $Z^u(\cdot)$ defined by the unique solution of the following mean-field SDE

$$\begin{cases} dZ^u(t) &= Z^u(t)h(t, x^u(t), \mathbb{E}[x(t)], u(t), \mathbb{E}[u(t)])dY(t), \\ Z^u(0) &= 1. \end{cases} \quad (1.5)$$

Define a new probability measure \mathbb{P}^u on (Ω, \mathcal{F}) by $d\mathbb{P}^u = Z^u(1)d\mathbb{P}$. Then from Girsanov's theorem and (1.2), $(W(\cdot), W^u(\cdot))$ is an \mathbb{R}^2 -valued standard Brownian motion defined in the new probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{0 \leq t \leq T}, \mathbb{P}^u)$. So $(\mathbb{P}^u, X^u(\cdot), Y(\cdot), W(\cdot), W^u(\cdot))$ is a weak solution on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \in \mathcal{T}})$ of (1.1) and (1.2).

Now for any given admissible control $u(\cdot) \in U_{ad}^W$ and the corresponding weak solution $(\mathbb{P}^u, x^u(\cdot), Y(\cdot), W(\cdot), W^u(\cdot))$ of (1.1) and (1.2), we introduce the following cost functional in the weak form,

$$\begin{aligned} J(u(\cdot)) = & \mathbb{E}^u \left[\int_0^T l(t, x(t), \mathbb{E}[x(t)], u(t), \mathbb{E}[u(t)])dt \right. \\ & \left. + m(X(T), \mathbb{E}[x(T)]) \right], \end{aligned} \quad (1.6)$$

where \mathbb{E}^u denotes the expectation with respect to the probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{0 \leq t \leq T}, \mathbb{P}^u)$ and $l : \mathcal{T} \times \Omega \times \mathbb{R}^n \times \mathbb{R}^n \times U \times U \rightarrow \mathbb{R}$, $m : \Omega \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$ are given random mappings satisfying the following assumption:

Assumption 1.2. l is $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}^n) \otimes \mathcal{B}(\mathbb{R}^n) \otimes \mathcal{B}(U) \otimes \mathcal{B}(U)$ -measurable, and m is $\mathcal{F}_T \otimes \mathcal{B}(\mathbb{R}^n) \otimes \mathcal{B}(\mathbb{R}^n)$ -measurable. For each $(x, y, u, v) \in \mathbb{R}^n \times \mathbb{R}^n \times U \times U$, $f(\cdot, x, y, u, v)$ is an \mathbb{F} -adapted process, and $m(x, y)$ is an \mathcal{F}_T -measurable random variable. For almost all $(t, \omega) \in [0, T] \times \Omega$, the mappings

$$(x, y, u, v) \rightarrow l(t, \omega, x, y, u, v)$$

and

$$(x, y) \rightarrow m(\omega, x, y)$$

are continuous differentiable with respect to (x, y, u, v) with appropriate growths, respectively. More precisely, there exists a constant $C > 0$ such that for all $x, y \in \mathbb{R}^n, u, v \in U$ and a.e. $(t, \omega) \in [0, T] \times \Omega$,

$$\begin{cases} (1 + |x| + |y| + |u| + |v|)^{-1} (|l_x(t, x, y, u, v)| + |l_y(t, x, y, u, v)| + |l_u(t, x, y, u, v)| + |l_v(t, x, y, u, v)|) \\ \quad + (1 + |x|^2 + |y|^2 + |u|^2 + |v|^2)^{-1} |l(t, x, y, u, v)| \leq C; \\ (1 + |x|^2 + |y|^2)^{-1} |m(x, y)| + (1 + |x| + |y|)^{-1} (|m_x(x, y)| + |m_y(x, y)|) \leq C. \end{cases}$$

Under Assumption 1.1 and 1.2, by the estimates (1.13) and (1.14), we get that

$$\begin{aligned} |J(u(\cdot))| &\leq K \mathbb{E} \left[\int_0^T |Z^u(t)| (1 + |x^u(t)|^2 + |\mathbb{E}[x^u(t)]|^2 + |u(t)|^2 + |\mathbb{E}[u(t)]|^2) dt \right] \\ &\leq K \left\{ \mathbb{E} \left[\sup_{t \in \mathcal{T}} |Z^u(t)|^2 \right] \right\}^{\frac{1}{2}} \left\{ \mathbb{E} \left[\sup_{t \in \mathcal{T}} |x(t)|^4 \right] + \mathbb{E} \left[\left(\int_0^T |u(t)|^2 dt \right)^2 \right] + 1 \right\}^{\frac{1}{2}} \\ &< \infty, \end{aligned} \quad (1.7)$$

which implies that the cost functional is well-defined.

Then we can put forward the following partially observed optimal control problem in its weak formulation, i.e., with changing the reference probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{0 \leq t \leq T}, \mathbb{P}^u)$, as follows.

Problem 1.1. Find an admissible control $\bar{u}(\cdot) \in U_{ad}^W$ such that

$$J(\bar{u}(\cdot)) = \inf_{u(\cdot) \in U_{ad}^W} J(u(\cdot)),$$

subject to the state equation (1.1), the observation equation (1.2) and the cost functional (1.6).

Obviously, according to Bayes' formula, the cost functional (1.6) can be rewritten as

$$\begin{aligned} J(u(\cdot)) &= \mathbb{E} \left[\int_0^T Z^u(t) l(t, x(t), \mathbb{E}[x(t)], u(t), \mathbb{E}[u(t)]) dt \right. \\ &\quad \left. + Z^u(T) m(x(T), \mathbb{E}[x(T)]) \right]. \end{aligned} \quad (1.8)$$

Therefore, we can translate Problem 1.1 into the following equivalent optimal control problem in its strong formulation, i.e., without changing the reference probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{0 \leq t \leq T}, \mathbb{P})$, where $Z^u(\cdot)$ will be regarded as an additional state process besides the state process $x^u(\cdot)$.

Problem 1.2. Find an admissible control $\bar{u}(\cdot)$ such that

$$J(\bar{u}(\cdot)) = \inf_{u(\cdot) \in U_{ad}^W} J(u(\cdot)),$$

subject to the cost functional (1.8) and the following state equation

$$\begin{cases} dx(t) &= [(b - \tilde{g}h)(t, x(t), \mathbb{E}[x(t)], u(t), \mathbb{E}[u(t)])]dt + g(t, x(t), \mathbb{E}[x(t)], u(t), \mathbb{E}[u(t)])dW(t) \\ &\quad + \tilde{g}(t, x(t), \mathbb{E}[x(t)], u(t), \mathbb{E}[u(t)])dY(t), \\ dZ(t) &= Z(t)h(t, x(t), \mathbb{E}[x(t)], u(t), \mathbb{E}[u(t)])dY(t), \\ Z(0) &= 1, \\ x(0) &= a \in \mathbb{R}^n. \end{cases} \quad (1.9)$$

Any $\bar{u}(\cdot) \in U_{ad}^W$ satisfying above is called an optimal control process of Problem 1.2 and the corresponding state process $(\bar{x}(\cdot), \bar{Z}(\cdot))$ is called the optimal state process. Correspondingly $(\bar{u}(\cdot); \bar{x}(\cdot), \bar{Z}(\cdot))$ is called an optimal pair of Problem 1.2.

Remark 1.2. The present formulation of the partially observed optimal control problem is quite similar to a completely observed optimal control problem; the only difference lies in the admissible class U_{ad}^W of controls.

In this paper, provided the original state equation (1.1) and the observation equation (1.2), we will also study the partially observed optimal control problem in its strong formulation, i.e. without changing the reference probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{0 \leq t \leq T}, \mathbb{P})$. Precisely, different from the cost functional (1.6), the cost functional in this case is defined by

$$J(u(\cdot)) = \mathbb{E} \left[\int_0^T l(t, X(t), \mathbb{E}[x(t)], u(t), \mathbb{E}[u(t)])dt + m(X(T), \mathbb{E}[x(T)]) \right]. \quad (1.10)$$

Note that $\mathbb{E}(\cdot)$ is the expectation with the original probability \mathbb{P} independent of the control $u(\cdot)$. In this case, different from the partially observed optimal control problem in weak sense discussed before, we do not need require the admissible control process satisfies (1.11). In this case, an admissible control process is defined as a $\{\mathcal{F}_t^Y\}_{0 \leq t \leq T}$ adapted stochastic process valued in U satisfying

$$\mathbb{E} \left[\int_0^T |u(t)|^2 dt \right] < \infty. \quad (1.11)$$

The set of all admissible controls in this case is denoted by U_{ad}^S .

Then we can put forward the partially observed optimal control problem in its strong formulation as follows.

Problem 1.3. Find an admissible control $\bar{u}(\cdot)$ such that

$$J(\bar{u}(\cdot)) = \inf_{u(\cdot) \in U_{ad}^W} J(u(\cdot)),$$

subject to the cost functional (1.10) and the following state equation

$$\begin{cases} dx(t) &= [(b - \tilde{g}h)(t, x(t), \mathbb{E}[x(t)], u(t), \mathbb{E}[u(t)])dt + g(t, x(t), \mathbb{E}[x(t)], u(t), \mathbb{E}[u(t)])dW(t) \\ &\quad + \tilde{g}(t, x(t), \mathbb{E}[x(t)], u(t), \mathbb{E}[u(t)])dY(t), \\ x(0) &= a \in \mathbb{R}^n. \end{cases} \quad (1.12)$$

Note that under Assumptions 1.1 and 1.2, for any admissible control $u(\cdot) \in U_{ad}^S$, by Lemma 1.4 below, the state (1.12) has a unique solution $x(\cdot) \in S_{\mathcal{F}}^2(0, T; \mathbb{R}^n)$ and $J(u(\cdot)) < \infty$, so Problem 1.3 is well-defined.

Before concluding this subsection, we give the well-posedness of the state equation as well as some useful estimates which can be showed easily by the classic compression mapping theorem combining with Gronwall's inequality and B-D-G inequality.

Lemma 1.4. *Let Assumption 1.1 holds. Then for any $u(\cdot) \in M_{\mathcal{F}}^{\beta}(0, T; L^2(0, T; \mathbb{R}^k))$, the state equation (1.9) admits a unique strong solution $(x(\cdot), Z(\cdot)) \in S_{\mathcal{F}}^{\beta}(0, T; \mathbb{R}^{n+1})$. Moreover, we have the following estimates:*

$$\mathbb{E} \left[\sup_{t \in \mathcal{T}} |x(t)|^{\beta} \right] \leq K \left\{ 1 + |a|^{\beta} + \mathbb{E} \left[\left(\int_0^T |u(t)|^2 dt \right)^{\frac{\beta}{2}} \right] \right\}, \quad (1.13)$$

and for any $\alpha \geq 2$,

$$\mathbb{E} \left[\sup_{t \in \mathcal{T}} |Z(t)|^{\alpha} \right] \leq K. \quad (1.14)$$

Further, if $(\bar{x}(\cdot), \bar{Z}(\cdot))$ is the unique strong solution corresponding to another $\bar{u}(\cdot) \in M_{\mathcal{F}}^{\beta}(0, T; L^2(0, T; \mathbb{R}^k))$, then the following estimate holds

$$\mathbb{E} \left[\sup_{t \in \mathcal{T}} |x(t) - \bar{x}(t)|^{\beta} \right] + \mathbb{E} \left[\sup_{t \in \mathcal{T}} |Z(t) - \bar{Z}(t)|^{\beta} \right] \leq K \mathbb{E} \left[\int_0^T |u(t) - \bar{u}(t)|^2 dt \right]^{\frac{\beta}{2}}. \quad (1.15)$$

1.3 Related Development and Contributions of this paper

Most recently, stochastic optimal control problems of stochastic differential equations (SDE) of mean-field type have attracted a great deal of attention due to its wide range of applications in economics and finance such as mean-variance portfolio selection problems. As stated by Djehiche and Tembine (2016), the main feature of this class of control problem is that the cost functional, the coefficients of the drift and diffusion terms of the state equation depend not only on the state and the control, but also on their probability distribution. The presence of the mean-field term makes the control problem become to be time-inconsistent so that the dynamic programming principle (DPP) does not work, which motivates to establish the stochastic maximum principle (SMP) to solve this type of optimal control problems instead of trying extensions of DPP. It is well-known that adjoint equations play a critical role in the formulation of the stochastic maximum principle. Intuitively speaking, the adjoint equation of a controlled state equation of mean-field type is a backward stochastic differential equation (BSDE) of mean -field type. So it is not until Buckdahn et al (2009a, 2009b) established the

results on the mean-field BSDEs that the stochastic maximum principle and related theoretical result and application for the optimal control system of mean-field type has become an important and popular topic. we refer to interested readers to Andersson and Djehiche (2011), Buckdahn et al (2011), Li (2012), Meyer-Brandis et al (2012), Shen and Siu (2013), Du et al(2013), Elliott (2013), Hafayed (2013), Yong(2013), Chala (2014), Shen et al(2014), Meng and Shen(2015) and the reference therein for the various optimal control theory results on the mean-field models with full observation.

A great of results on stochastic optimal control without mean-field term under partial observation or partial information have been obtained by many authors for various types of stochastic systems via establishing the corresponding MP and DPP. See e.g., Bensoussan (1983), Tang (1998), Baghery et al. (2007), Wu (2010), Wang and Wu (2009), Wang et al (2013, 2015a), and the reference therein for more detailed discussion.

The purpose of this paper is an extension to the optimal control of stochastic diffusion of mean-field type under partial observation (see Problem 1.1, 1.3). Along this topic, due to the theoretical and practical interest, recently, it become more popular, e.g, Wang et al (2014a, 2014b, 2015b, 2016), Djehiche and Tembine (2016), Ma and Liu (2017), where the corresponding maximum principles are established and practical finance applications are illustrated. Different from the above mentioned references, for our optimal control problem of mean-field type, there are some distinctive features and contribution worthy of being emphasizing. First, our state system is a stochastic nonlinear system where the observation noise $W^u(\cdot)$ is allowed to enter into our state equation and the observation coefficients may depend not only on the control process, but also on its probability distribution. Therefore, our model is more general and complicated, which leads to that our adjoint equation is more different and the derivation of our main result need more skills required. Second, for Problem 1.1 in weak formulation, under the standard assumption on the coefficients in which case the linear quadratic optimal control problem is included, the required integral condition for our admissible control $u(\cdot)$ is

$$\mathbb{E}\left[\left(\int_0^T |u(t)|^2 dt\right)^2\right] < \infty, \quad (1.16)$$

which is more weaker than that in the existed reference,(cf., for example, see, Wang et al (2014a, 2014b, 2016)) where the required integral condition for their admissible control $u(\cdot)$ is

$$\sup_{t \in \mathcal{T}} \mathbb{E}\left[|u(t)|^8 dt\right] < \infty. \quad (1.17)$$

When we require all the coefficients involved in the state equation and the cost functional are bounded (see, for example, Djehiche and Tembine (2016)), the integral condition can be weakened to the following

$$\mathbb{E}\left[\int_0^T |u(t)|^2 dt\right] < \infty, \quad (1.18)$$

but in this case, the classic LQ problem are not included. Under (1.16), our main result on the stochastic maximum principle can be obtained based on the refined estimate (1.13)- (1.15) for the state equation. Note that for Problem 1.3 in the strong formulation, we need only require that admissible control satisfied (1.18) because the stochastic process $Z(\cdot)$ (see (1.5)) is not involved in the cost functional (see, for example, Wang et al (2015a), Ma and Liu (2017)). Third, the main contribution of this paper is that the corresponding maximum principle for the

partial observed optimal control is established under our stochastic model of mean-field type by establishing a variation formula of the cost functional. The main idea is to get directly a variation formula in terms of the Hamiltonian and the associated adjoint system which is a linear backward stochastic differential equation of mean-field and neither the variational equation nor the corresponding Taylor type expansions of the cost functional and the state process will be introduced. As an application, the LQ problem of mean-field type under partial observation is illustrated and solved by the stochastic maximum principle. This paper can be regarded as an addition to the study of partially observed stochastic optimal control problems of mean-field type.

The rest of this paper is organized as follows. In section 2, the necessary maximum principle in a weak formulation is established by convex variation and adjoint calculation. Section 3 is devoted to deriving necessary as well as sufficient optimality conditions for Problem 1.3 in a strong formulation in the form of stochastic maximum principles in a unified way. As an application, a partially observed LQ problem of mean-field type is studied detailed and the corresponding dual characterization and state feed-back presentation of the optimal control are obtained by the stochastic maximum principles established in section 3 and the classic technique of completing squares, respectively.

2 Stochastic Maximum Principle in Weak Formulation

This section is devoted to establishing the stochastic maximum principle of Problem 1.1 or Problem 1.2, i.e., establishing the necessary optimality condition of Pontryagin's type for an admissible control to be optimal. To this end, for the state equation (1.9), we first introduce the corresponding adjoint equation. Actually, define the Hamiltonian function $H : [0, T] \times \Omega \times \mathbb{R}^n \times \mathbb{R}^n \times U \times U \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ by

$$\begin{aligned} H(t, x, y, u, v, p, q, \tilde{q}, \tilde{R}) \\ = \langle p, b(t, x, y, u, v) \rangle + \langle q, g(t, x, y, u, v) \rangle + \langle \tilde{q}, \tilde{g}(t, x, y, u, v) \rangle + \tilde{R}h(t, x, y, u, v) + l(t, x, y, u, v). \end{aligned} \quad (2.1)$$

For the state equation (1.9) associated with any given admissible pair $(\bar{u}(\cdot), \bar{x}(\cdot), \bar{Z}(\cdot))$, the corresponding adjoint equation is defined as follows:

$$\left\{ \begin{aligned} d\bar{r}(t) &= -l(t, \bar{x}(t), \mathbb{E}[\bar{x}(t)], \bar{u}(t), \mathbb{E}[\bar{u}(t)])dt + \bar{R}(t) dW(t) + \bar{\tilde{R}}(t) dW^{\bar{u}}(t), \\ d\bar{p}(t) &= -\left\{ H_x(t, \bar{x}(t), \mathbb{E}[\bar{x}(t)], \bar{u}(t), \mathbb{E}[\bar{u}(t)]) + \frac{1}{\bar{Z}(t)} \mathbb{E}^{\bar{u}}[H_y(t, \bar{x}(t), \mathbb{E}[\bar{x}(t)], \bar{u}(t), \mathbb{E}[\bar{u}(t)])] \right\} dt \\ &\quad + \bar{q}(t) dW(t) + \bar{\tilde{q}}(t) dW^{\bar{u}}(t), \\ \bar{r}(T) &= m(\bar{x}(T), \mathbb{E}[\bar{x}(T)]), \\ \bar{p}(T) &= m_x(\bar{x}(T), \mathbb{E}[\bar{x}(T)]) + \frac{1}{\bar{Z}(T)} \mathbb{E}^{\bar{u}}[m_x(\bar{x}(T), \mathbb{E}[\bar{x}(T)])], \end{aligned} \right. \quad (2.2)$$

where

$$H(t, x, y, u, v) =: H(t, x, y, u, v, \bar{p}(t), \bar{q}(t), \bar{\tilde{q}}(t), \bar{R}(t) - \tilde{g}(t, \bar{x}(t), \mathbb{E}[\bar{x}(t)], \bar{u}(t), \mathbb{E}[\bar{u}(t)])^\top \bar{p}(t)). \quad (2.3)$$

Note the adjoint equation (2.2) is a mean-field backward stochastic differential equation whose solution consists of an 6-tuple process $(\bar{p}(\cdot), \bar{q}(\cdot), \bar{\tilde{q}}(\cdot), \bar{r}(\cdot), \bar{R}(\cdot), \bar{\tilde{R}}(\cdot))$. Under Assumptions 1.1 and 1.2, by Buckdahn (2009b), it is easily to see that the adjoint equation (2.2) admits a unique solution $(\bar{p}(\cdot), \bar{q}(\cdot), \bar{\tilde{q}}(\cdot), \bar{r}(\cdot), \bar{R}(\cdot), \bar{\tilde{R}}(\cdot)) \in S_{\mathcal{F}}^2(0, T; \mathbb{R}^n) \times M_{\mathcal{F}}^2(0, T; \mathbb{R}^n) \times M_{\mathcal{F}}^2(0, T; \mathbb{R}^n) \times S_{\mathcal{F}}^2(0, T; \mathbb{R}) \times M_{\mathcal{F}}^2(0, T; \mathbb{R}) \times M_{\mathcal{F}}^2(0, T; \mathbb{R})$, also called the adjoint process corresponding the admissible pair $(\bar{u}(\cdot); \bar{x}(\cdot), \bar{Z}(\cdot))$.

Now we are in a position to state our main result: stochastic maximum principle of Problem 1.1 or 1.2.

Theorem 2.1. *Let assumptions 1.1 and 1.2 be satisfied. Let $(\bar{u}(\cdot); \bar{x}(\cdot), \bar{Z}(\cdot))$ be an optimal pair of Problem 1.2 associated with the adjoint process $(\bar{p}(\cdot), \bar{q}(\cdot), \bar{\tilde{q}}(\cdot), \bar{r}(\cdot), \bar{R}(\cdot), \bar{\tilde{R}}(\cdot))$. Then the optimality condition*

$$\left\langle \mathbb{E}[\bar{Z}(t)\bar{H}_u(t)|\mathcal{F}_t^Y] + \mathbb{E}^{\bar{u}}[\bar{H}_v(t)], u - \bar{u}(t) \right\rangle \geq 0 \quad (2.4)$$

holds for any $u \in U$ and a.e. $(t, \omega) \in [0, T] \times \Omega$. Here using the notation (2.3), we set

$$\bar{H}_u(t) = H_u(t, x(t), \mathbb{E}[\bar{x}(t)], u(t), \mathbb{E}[u(t)]), \quad (2.5)$$

and

$$\bar{H}_v(t) = H_v(t, x(t), \mathbb{E}[\bar{x}(t)], u(t), \mathbb{E}[u(t)]). \quad (2.6)$$

To prove this theorem, we first need to establish the variation formula for the cost functional (1.6) or (1.8) by the classical convex variation method and dual technique.

Since the control domain U is convex, for any given admissible control $\bar{u}(\cdot), u(\cdot) \in U_{ad}^W$, the following perturbed control process $u^\epsilon(\cdot)$:

$$u^\epsilon(\cdot) = \bar{u}(\cdot) + \epsilon(u(\cdot) - \bar{u}(\cdot)) , \quad 0 \leq \epsilon \leq 1 , \quad (2.7)$$

is also an element of U_{ad}^W . We denote by $(x^\epsilon(\cdot), Z^\epsilon(\cdot))$ the solution to the state equation (1.9) corresponding to $u^\epsilon(\cdot)$. To unburden our notation, we will use the following abbreviations:

$$\begin{cases} m^\epsilon(T) = m(x^\epsilon(T), \mathbb{E}[x^\epsilon(T)]) , \bar{m}(T) = m(\bar{x}(T), \mathbb{E}[\bar{x}(T)]) , \\ \phi^\epsilon(t) = \phi(t, x^\epsilon(t), \mathbb{E}[x^\epsilon(t)], u^\epsilon(t), \mathbb{E}[u^\epsilon(t)]) , \quad \phi = b, g, \tilde{g}, h, l , \\ \bar{\phi}(t) = \phi(t, \bar{x}(t), \mathbb{E}[\bar{x}(t)], \bar{u}(t), \mathbb{E}[\bar{u}(t)]) , \quad \phi = b, g, \tilde{g}, h, l , \\ H^\epsilon(t) = H(t, x^\epsilon(t), \mathbb{E}[x^\epsilon(t)], u^\epsilon(t), \mathbb{E}[u^\epsilon(t)]), \\ \bar{H}(t) = H(t, \bar{x}(t), \mathbb{E}[\bar{x}(t)], \bar{u}(t), \mathbb{E}[\bar{u}(t)]), \\ \bar{H}_x(t) = H_x(t, \bar{x}(t), \mathbb{E}[\bar{x}(t)], \bar{u}(t), \mathbb{E}[\bar{u}(t)]), \\ \bar{H}_y(t) = H_y(t, \bar{x}(t), \mathbb{E}[\bar{x}(t)], \bar{u}(t), \mathbb{E}[\bar{u}(t)]) . \end{cases} \quad (2.8)$$

To establish the variation formula for the cost function (1.6) or (1.8), we need the following two basic Lemmas.

Lemma 2.2. *Let Assumptions 1.1 and 1.2 be satisfied. Then for any $2 \leq \gamma \leq 4$, we have*

$$\mathbb{E} \left[\sup_{t \in \mathcal{T}} |x^\epsilon(t) - \bar{x}(t)|^\gamma \right] + \mathbb{E} \left[\sup_{t \in \mathcal{T}} |Z^\epsilon(t) - \bar{Z}(t)|^\gamma \right] = O(\epsilon^\gamma) . \quad (2.9)$$

Proof. By the estimate (1.15) in Lemma 1.4 and the definition of $u^\epsilon(\cdot)$ (see (2.7)), we have

$$\begin{aligned}
& \mathbb{E} \left[\sup_{t \in \mathcal{T}} |x^\epsilon(t) - \bar{x}(t)|^\gamma \right] + \mathbb{E} \left[\sup_{t \in \mathcal{T}} |Z^\epsilon(t) - \bar{Z}(t)|^\gamma \right] \\
& \leq K \mathbb{E} \left[\int_0^T |u^\epsilon(t) - \bar{u}(t)|^2 dt \right]^{\frac{\gamma}{2}} \\
& = K \epsilon^\gamma \mathbb{E} \left[\int_0^T |u(t) - \bar{u}(t)|^2 dt \right]^{\frac{\gamma}{2}} \\
& = O(\epsilon^\gamma).
\end{aligned} \tag{2.10}$$

The proof is complete. \square

Next we represent the difference $J(u^\epsilon(\cdot)) - J(\bar{u}(\cdot))$ in terms of the Hamiltonian H and the adjoint process $(\bar{p}(\cdot), \bar{q}(\cdot), \bar{\tilde{q}}(\cdot), \bar{r}(\cdot), \bar{R}(\cdot), \bar{\tilde{R}}(\cdot))$ as well as other relevant expressions.

Lemma 2.3. *Let Assumptions 1.1 and 1.2 be satisfied. Using the notations (2.3) and (2.8), we have*

$$\begin{aligned}
& J(u^\epsilon(\cdot)) - J(\bar{u}(\cdot)) \\
& = \mathbb{E}^{\bar{u}} \left[\int_0^T (H^\epsilon(t) - \bar{H}(t) - \langle x^\epsilon(t) - \bar{x}(t), \bar{H}_x(t) + \frac{1}{\bar{Z}(t)} \mathbb{E}^{\bar{u}}[\bar{H}_y(t)] \rangle) dt \right] \\
& - \mathbb{E}^{\bar{u}} \left[\int_0^T \langle (\tilde{g}^\epsilon(t) - \bar{\tilde{g}}(t))(h^\epsilon(t) - \bar{h}(t)), \bar{p}(t) \rangle dt \right] + \mathbb{E} \left[\int_0^T (Z^\epsilon(t) - \bar{Z}(t))(l^\epsilon(t) - \bar{l}(t)) dt \right] \\
& + \mathbb{E} \left[\int_0^T \bar{\tilde{R}}(t)(Z^\epsilon(t) - \bar{Z}(t))(h^\epsilon(t) - \bar{h}(t)) dt \right] + \mathbb{E} \left[(Z^\epsilon(T) - \bar{Z}(T))(m^\epsilon(T) - \bar{m}(T)) \right] \\
& + \mathbb{E}^u \left[m^\epsilon(T) - \bar{m}(T) - \left\langle x^\epsilon(T) - \bar{x}(T), \bar{m}_x(T) + \frac{1}{\bar{Z}(T)} \mathbb{E}^{\bar{u}}[\bar{m}_x(T)] \right\rangle \right],
\end{aligned} \tag{2.11}$$

for any $u(\cdot), u^\epsilon(\cdot) \in \mathcal{A}$ and $\epsilon \in [0, 1]$.

Proof. From the definitions of the Hamiltonian H (see (2.1)) and the cost functional $J(u(\cdot))$ (see (1.6) or (1.8)), it is easy to check that

$$\begin{aligned}
& J(u^\epsilon(\cdot)) - J(\bar{u}(\cdot)) \\
& = \mathbb{E} \left[\int_0^T Z^\epsilon(t) l^\epsilon(t) dt + Z^\epsilon(T) m^\epsilon(T) \right] - \mathbb{E} \left[\int_0^T \bar{Z}(t) \bar{l}(t) dt + \bar{Z}(T) \bar{m}(T) \right] \\
& = \mathbb{E}^{\bar{u}} \left[\int_0^T (l^\epsilon(t) - \bar{l}(t)) dt \right] + \mathbb{E}^{\bar{u}}[m^\epsilon(T) - \bar{m}(T)] \\
& + \mathbb{E} \left[\int_0^T l^\epsilon(t) (\bar{Z}^\epsilon(t) - \bar{Z}(t)) dt \right] + \mathbb{E}[m^\epsilon(T) (\bar{Z}^\epsilon(T) - \bar{Z}(T))]. \\
& = \mathbb{E}^{\bar{u}} \left[\int_0^T \left(H^\epsilon(t) - \bar{H}(t) - \langle b^\epsilon(t) - \bar{b}(t), p(t) \rangle - \langle g^\epsilon(t) - \bar{g}(t), q(t) \rangle - \langle \tilde{g}^\epsilon(t) - \bar{\tilde{g}}(t), \tilde{q}(t) \rangle \right. \right. \\
& \quad \left. \left. - \langle h^\epsilon(t) - \bar{h}(t), \tilde{R}(t) - \bar{\tilde{g}}^\top(t) p(t) \rangle \right) dt \right] + \mathbb{E}^{\bar{u}}[m^\epsilon(T) - \bar{m}(T)] \\
& + \mathbb{E} \left[\int_0^T l^\epsilon(t) (\bar{Z}^\epsilon(t) - \bar{Z}(t)) dt \right] + \mathbb{E}[m^\epsilon(T) (\bar{Z}^\epsilon(T) - \bar{Z}(T))].
\end{aligned} \tag{2.12}$$

From (1.1) and the relation (1.2), it is easily to see that $x^\varepsilon(\cdot) - x(\cdot)$ satisfies the following mean-field SDE

$$\begin{cases} d(x^\varepsilon(t) - x(t)) = [b^\varepsilon(t) - \bar{b}(t)]dt + [g^\varepsilon(t) - \bar{g}(t)]dW(t) + [\tilde{g}^\varepsilon(t) - \tilde{\bar{g}}(t)]dW^{\bar{u}}(t) \\ \quad - \tilde{g}^\varepsilon(t)[h^\varepsilon(t) - \bar{h}(t)]dt, \\ x^\varepsilon(0) - \bar{x}(0) = 0. \end{cases} \quad (2.13)$$

From (2.2), we know that $(\bar{p}(\cdot), \bar{q}(\cdot), \tilde{\bar{q}}(\cdot))$ satisfies the following mean-field BSDE

$$\begin{cases} d\bar{p}(t) = -\left\{ \bar{H}_x(t) + \frac{1}{\bar{Z}(t)}\mathbb{E}^{\bar{u}}[\bar{H}_y(t)] \right\}dt + \bar{q}(t)dW(t) + \tilde{\bar{q}}(t)dW^{\bar{u}}(t), \\ \bar{p}(T) = \bar{m}_x(T) + \frac{1}{\bar{Z}(T)}\mathbb{E}^{\bar{u}}[\bar{m}_y(T)]. \end{cases} \quad (2.14)$$

Applying Itô's formula to $\langle x^\varepsilon(t) - \bar{x}(t), \bar{p}(t) \rangle$ and taking expectation under the probability $\mathbb{P}^{\bar{u}}$ results in

$$\begin{aligned} & \mathbb{E}^{\bar{u}} \left[\int_0^T \langle b^\varepsilon(t) - \bar{b}(t), p(t) \rangle + \langle g^\varepsilon(t) - \bar{g}(t), q(t) \rangle + \langle \tilde{g}^\varepsilon(t) - \tilde{\bar{g}}(t), \tilde{q}(t) \rangle dt \right] \\ = & \mathbb{E}^{\bar{u}} \left[\int_0^T \left\langle x^\varepsilon(t) - \bar{x}(t), H_x(t) + \frac{1}{\bar{Z}(t)}\mathbb{E}^{\bar{u}}[H_y(t)] \right\rangle dt \right] \end{aligned} \quad (2.15)$$

$$\begin{aligned} & + \mathbb{E}^{\bar{u}} \left[\left\langle x^\varepsilon(T) - \bar{x}(T), \bar{m}_x(T) + \frac{1}{\bar{Z}(T)}\mathbb{E}^{\bar{u}}[\bar{m}_y(T)] \right\rangle \right] \\ & + \mathbb{E}^{\bar{u}} \left[\int_0^T \left\langle (h^\varepsilon(t) - \bar{h}(t))\tilde{g}^\varepsilon(t), p(t) \right\rangle dt \right]. \end{aligned} \quad (2.16)$$

On the other hand, from (1.5), it is easy to check that $Z^\varepsilon(\cdot) - \bar{Z}(\cdot)$ satisfies the following mean-field SDE

$$\begin{cases} d(Z^\varepsilon(t) - \bar{Z}(t)) = (Z^\varepsilon(t)h^\varepsilon(t) - \bar{Z}(t)\bar{h}(s))dY(t), \\ Z^\varepsilon(0) - \bar{Z}(0) = 0, \end{cases} \quad (2.17)$$

and from (2.2), $(\bar{r}(\cdot), \bar{R}(\cdot), \tilde{\bar{R}}(\cdot))$ satisfies the following mean-field BSDE

$$\begin{cases} d\bar{r}(t) = -\bar{l}(t)dt + \bar{R}(t)dW(t) + \tilde{\bar{R}}(t)dY(t) - \tilde{\bar{R}}(t)\bar{h}(t)dt, \\ \bar{r}(T) = \bar{m}(T). \end{cases} \quad (2.18)$$

Applying Itô's formula to $(Z^\varepsilon(t) - \bar{Z}(t))\bar{r}(t)$ and taking expectation under the probability P results in

$$\begin{aligned} & \mathbb{E}[(Z^\varepsilon(T) - \bar{Z}(T))\bar{m}(T)] + \mathbb{E} \left[\int_0^T \bar{l}(t)(\bar{Z}^\varepsilon(t) - \bar{Z}(t))dt \right] \\ = & \mathbb{E} \left[\int_0^T \tilde{\bar{R}}(t)Z^\varepsilon(t)(h^\varepsilon(t) - \bar{h}(s))dt \right]. \end{aligned} \quad (2.19)$$

Now putting (2.15) and (2.19) into (2.12), we deduce that (2.11) holds. The proof is complete. \square

Now we are in the position to use Lemma 2.2 and Lemma 2.3 to derive the variational formula for the cost functional $J(u(\cdot))$ in terms of the Hamiltonian H .

Theorem 2.4. *Let Assumptions 1.1 and 1.2 be satisfied. Let $(\bar{u}(\cdot), \bar{x}(\cdot), \bar{Z}(\cdot))$ and $(u(\cdot), x(\cdot), Z(\cdot))$ be two any given admissible pair. And let $(\bar{p}(\cdot), \bar{q}(\cdot), \bar{\tilde{q}}(\cdot), \bar{r}(\cdot), \bar{R}(\cdot), \bar{\tilde{R}}(\cdot))$ be the adjoint process corresponding to the admissible pair $(\bar{u}(\cdot), \bar{x}(\cdot), \bar{Z}(\cdot))$. Then for the cost functional (1.6) or (1.8), using the notations (2.5), (2.6) and (2.8), we have the following variation formula:*

$$\begin{aligned} \frac{d}{d\epsilon} J(\bar{u}(\cdot) + \epsilon(u(\cdot) - \bar{u}(\cdot)))|_{\epsilon=0} &:= \lim_{\epsilon \rightarrow 0^+} \frac{J(\bar{u}(\cdot) + \epsilon(u(\cdot) - \bar{u}(\cdot))) - J(\bar{u}(\cdot))}{\epsilon} \\ &= \mathbb{E} \left[\int_0^T \langle \bar{Z}(t) \bar{H}_u(t) + \mathbb{E}^{\bar{u}}[\bar{H}_v(t)], u(t) - \bar{u}(t) \rangle dt \right]. \end{aligned} \quad (2.20)$$

Proof. Set

$$u^\epsilon(\cdot) = \bar{u}(\cdot) + \epsilon(u(\cdot) - \bar{u}(\cdot)), \quad 0 \leq \epsilon \leq 1. \quad (2.21)$$

Let $(x^\epsilon(\cdot), Z^\epsilon(\cdot))$ be the state process corresponding to $u^\epsilon(\cdot)$. For notational simplicity, using the notations (2.5), (2.6) and (2.8), we write

$$\begin{aligned} \beta_1^\epsilon &:= \mathbb{E}^{\bar{u}} \left[\int_0^T \left(H^\epsilon(t) - \bar{H}(t) - \langle x^\epsilon(t) - \bar{x}(t), \bar{H}_x(t) + \frac{1}{\bar{Z}(t)} \mathbb{E}^{\bar{u}}[\bar{H}_y(t)] \rangle \right. \right. \\ &\quad \left. \left. - \langle u^\epsilon(t) - \bar{u}(t), \bar{H}_u(t) + \frac{1}{\bar{Z}(t)} \mathbb{E}^{\bar{u}}[\bar{H}_v(t)] \rangle \right) dt \right], \end{aligned} \quad (2.22)$$

$$\beta_2^\epsilon := \mathbb{E}^{\bar{u}} \left[\int_0^T \langle (\tilde{g}^\epsilon(t) - \tilde{g}(t))(h^\epsilon(t) - \bar{h}(t)), \bar{p}(t) \rangle dt \right], \quad (2.23)$$

$$\beta_3^\epsilon := \mathbb{E}^{\bar{u}} \left[m^\epsilon(T) - \bar{m}(T) - \left\langle x^\epsilon(T) - \bar{x}(T), \bar{m}_x(T) + \frac{1}{\bar{Z}(T)} \mathbb{E}^{\bar{u}}[\bar{m}_x(T)] \right\rangle \right], \quad (2.24)$$

$$\beta_4^\epsilon := \mathbb{E} \left[\int_0^T (\bar{Z}^\epsilon(t) - \bar{Z}(t))(l^\epsilon(t) - \bar{l}(t)) dt \right], \quad (2.25)$$

$$\beta_5^\epsilon := \mathbb{E} \left[\int_0^T \tilde{R}(t)(Z^\epsilon(t) - \bar{Z}(t))(h^\epsilon(t) - \bar{h}(t)) dt \right], \quad (2.26)$$

$$\beta_6^\epsilon := \mathbb{E} \left[(Z^\epsilon(T) - \bar{Z}(T))(m^\epsilon(T) - \bar{m}(T)) \right]. \quad (2.27)$$

By Lemma 2.3, we have

$$\begin{aligned} J(u^\epsilon(\cdot)) - J(\bar{u}(\cdot)) \\ = \beta^\epsilon + \epsilon \mathbb{E}^{\bar{u}} \left[\int_0^T \left\langle \bar{H}_u(t) + \frac{1}{\bar{Z}(t)} \mathbb{E}^{\bar{u}}[\bar{H}_v(t)], u(t) - \bar{u}(t) \right\rangle dt \right], \end{aligned} \quad (2.28)$$

where

$$\beta^\varepsilon = \beta_1^\varepsilon + \beta_2^\varepsilon + \beta_3^\varepsilon + \beta_4^\varepsilon + \beta_5^\varepsilon + \beta_6^\varepsilon. \quad (2.29)$$

Now we begin to prove

$$\beta^\varepsilon = o(\varepsilon) \quad (2.30)$$

Indeed, for β_2^ε , under Assumptions 1.1, we have

$$\begin{aligned} |\beta_2^\varepsilon| &\leq \mathbb{E} \left[\int_0^T |\tilde{g}^\varepsilon(t) - \tilde{g}(t)| |(h^\varepsilon(t) - \bar{h}(t))| |\bar{p}(t)| dt \right] \\ &\leq C \mathbb{E} \left[\int_0^T (|u^\varepsilon(t) - \bar{u}(t)| + |\mathbb{E}[u^\varepsilon(t)] - \mathbb{E}[\bar{u}(t)]| + |x^\varepsilon(t) - \bar{x}(t)| + |\mathbb{E}[x^\varepsilon(t)] - \mathbb{E}[\bar{x}(t)]|) \right. \\ &\quad \left. \cdot |(h^\varepsilon(t) - \bar{h}(t))| |\bar{p}(t)| dt \right] \\ &\leq C \left\{ \mathbb{E} \left[\int_0^T (|u^\varepsilon(t) - \bar{u}(t)|^2 + |x^\varepsilon(t) - \bar{x}(t)|^2) dt \right] \right\}^{\frac{1}{2}} \left\{ \mathbb{E} \left[\int_0^T |(h^\varepsilon(t) - \bar{h}(t))|^2 |\bar{p}(t)|^2 dt \right] \right\}^{\frac{1}{2}} \\ &\leq C \varepsilon \left\{ \int_0^T |(h^\varepsilon(t) - \bar{h}(t))|^2 |\bar{p}(t)|^2 dt \right\}^{\frac{1}{2}} \\ &= o(\varepsilon), \end{aligned} \quad (2.31)$$

where the last second inequality can be obtained by Lemma 2.2 and the last inequality can be got by the fact that

$$\lim_{\varepsilon \rightarrow 0} \mathbb{E} \left[\int_0^T |(h^\varepsilon(t) - \bar{h}(t))|^2 |\bar{p}(t)|^2 dt \right] = 0, \quad (2.32)$$

which can be obtained by the Lemma 2.2 and the dominated convergence theorem, since the function h is bounded.

For β_5 , in view of Lemma 2.2 and the dominated convergence theorem. we have

$$\begin{aligned} |\beta_5^\varepsilon| &\leq \mathbb{E} \left[\int_0^T |\tilde{R}(t)| |Z^\varepsilon(t) - \bar{Z}(t)| |h^\varepsilon(t) - \bar{h}(t)| dt \right] \\ &\leq \mathbb{E} \left[\sup_{0 \leq t \leq T} |Z^\varepsilon(t) - \bar{Z}(t)| \int_0^T |\tilde{R}(t)| |h^\varepsilon(t) - \bar{h}(t)| dt \right] \\ &\leq C \left\{ \mathbb{E} \left[\sup_{0 \leq t \leq T} |Z^\varepsilon(t) - \bar{Z}(t)|^2 \right] \right\}^{\frac{1}{2}} \left\{ \mathbb{E} \left[\int_0^T |\tilde{R}(t)|^2 |h^\varepsilon(t) - \bar{h}(t)|^2 dt \right] \right\}^{\frac{1}{2}} \\ &\leq C \varepsilon \left\{ \mathbb{E} \left[\int_0^T |\tilde{R}(t)|^2 |h^\varepsilon(t) - \bar{h}(t)|^2 dt \right] \right\}^{\frac{1}{2}} \\ &= o(\varepsilon). \end{aligned} \quad (2.33)$$

For β_4^ϵ , in view of Lemma 2.2 and the dominated convergence theorem, we have

$$\begin{aligned}
|\beta_4^\epsilon| &\leq \mathbb{E} \left[\int_0^T |\bar{Z}^\epsilon(t) - \bar{Z}(t)| |l^\epsilon(t) - \bar{l}(t)| dt \right] \\
&\leq \mathbb{E} \left[\sup_{0 \leq t \leq T} |Z^\epsilon(t) - \bar{Z}(t)| \int_0^T |l^\epsilon(t) - \bar{l}(t)| dt \right] \\
&\leq \left\{ \mathbb{E} \left[\sup_{0 \leq t \leq T} |Z^\epsilon(t) - \bar{Z}(t)|^2 \right] \right\}^{\frac{1}{2}} \left\{ \mathbb{E} \left[\int_0^T |l^\epsilon(t) - \bar{l}(t)|^2 dt \right] \right\}^{\frac{1}{2}} \\
&\leq C\epsilon \left\{ \mathbb{E} \left[\int_0^T |l^\epsilon(t) - \bar{l}(t)|^2 dt \right] \right\}^{\frac{1}{2}} \\
&= o(\epsilon).
\end{aligned} \tag{2.34}$$

For β_6^ϵ , in view of Lemma 2.2 and the dominated convergence theorem, we get

$$\begin{aligned}
|\beta_6^\epsilon| &\leq \mathbb{E} \left[|\bar{Z}^\epsilon(T) - \bar{Z}(T)| |m^\epsilon(T) - \bar{m}(T)| \right] \\
&\leq \left\{ \mathbb{E} \left[|Z^\epsilon(T) - \bar{Z}(T)|^2 \right] \right\}^{\frac{1}{2}} \left\{ \mathbb{E} \left[|m^\epsilon(T) - \bar{m}(T)|^2 \right] \right\}^{\frac{1}{2}} \\
&\leq C\epsilon \left\{ \mathbb{E} \left[|m^\epsilon(T) - \bar{m}(T)|^2 \right] \right\}^{\frac{1}{2}} \\
&= o(\epsilon).
\end{aligned} \tag{2.35}$$

For β_3^ϵ , under Assumptions 1.1 and 1.2, using the Taylor Expansions on the function m with respect to x and y , Lemma 2.2 and the dominated convergence theorem leads to

$$\begin{aligned}
|\beta_3^\epsilon| &\leq \mathbb{E}^{\bar{u}} \left[|\langle x^\epsilon(T) - \bar{x}(T), m_x^{\epsilon,\lambda}(T) - \bar{m}_x(T) + \frac{1}{\bar{Z}(T)} \mathbb{E}^{\bar{u}}[m_y^{\epsilon,\lambda}(T)] - \mathbb{E}[\bar{m}_y(T)] \rangle| \right] \\
&\leq \left\{ \mathbb{E} [|\bar{Z}(T)|^2] \right\}^{\frac{1}{2}} \left\{ \mathbb{E} [|x^\epsilon(T) - \bar{x}(T)|^4] \right\}^{\frac{1}{4}} \left\{ \mathbb{E} [|m_x^{\epsilon,\lambda}(T) - \bar{m}_x(T) + \mathbb{E}[m_y^{\epsilon,\lambda}(T)] - \mathbb{E}[\bar{m}_y(T)]|^4] \right\}^{\frac{1}{4}} \\
&\leq C\epsilon \left\{ \mathbb{E} [|m_x^{\epsilon,\lambda}(T) - \bar{m}_x(T) + \mathbb{E}[m_y^{\epsilon,\lambda}(T)] - \mathbb{E}[\bar{m}_y(T)]|^4] \right\}^{\frac{1}{4}} \\
&= o(\epsilon),
\end{aligned} \tag{2.36}$$

where we have used the following shorthand notations:

$$m_x^{\epsilon,\lambda}(T) = \int_0^1 m_x(\bar{x}(T) + \lambda(x^\epsilon(T) - \bar{x}(T)), \mathbb{E}[\bar{x}(T)] + \lambda(\mathbb{E}[x^\epsilon(T)] - \mathbb{E}[\bar{x}(T)])) d\lambda,$$

and

$$m_y^{\epsilon,\lambda}(T) = \int_0^1 m_y(\bar{x}(T) + \lambda(x^\epsilon(T) - \bar{x}(T)), \mathbb{E}[\bar{x}(T)] + \lambda(\mathbb{E}[x^\epsilon(T)] - \mathbb{E}[\bar{x}(T)])) d\lambda.$$

Similar to (2.36), using the Taylor Expansions on the function H with respect to x, y, u and v , Lemma 2.2 and the dominated convergence theorem, we have

$$\beta_1^\varepsilon = o(\varepsilon). \quad (2.37)$$

Therefore, combining (2.33)-(2.37) and using (2.29), we get that (2.30) holds. Then putting (2.30) into (2.28), we have

$$\begin{aligned} & \frac{d}{d\varepsilon} J(\bar{u}(\cdot) + \varepsilon(u(\cdot) - \bar{u}(\cdot)))|_{\varepsilon=0} \\ &= \lim_{\varepsilon \rightarrow 0^+} \frac{J(\bar{u}(\cdot) + \varepsilon(u(\cdot) - \bar{u}(\cdot))) - J(\bar{u}(\cdot))}{\varepsilon} \\ &= \lim_{\varepsilon \rightarrow 0^+} \frac{\beta^\varepsilon + \varepsilon \mathbb{E}^{\bar{u}} \left[\int_0^T \langle \bar{H}_u(t) + \mathbb{E}[\bar{H}_v(t)], u(t) - \bar{u}(t) \rangle dt \right]}{\varepsilon} \\ &= \mathbb{E} \left[\int_0^T \langle \bar{Z}(t) \bar{H}_u(t) + \mathbb{E}^{\bar{u}}[\bar{H}_v(t)], u(t) - \bar{u}(t) \rangle dt \right]. \end{aligned}$$

The proof is complete. \square

Now we are ready to prove Theorem 2.1

Proof. Since all admissible controls are $\{\mathcal{F}_t^Y\}_{t \in \mathcal{T}}$ -adapted processes, from the property of conditional expectation, Theorem 2.4 and the optimality of $\bar{u}(\cdot)$, we deduce that

$$\begin{aligned} & \mathbb{E} \left[\int_0^T \langle \mathbb{E}[\bar{Z}(t) \bar{H}_u(t) + \mathbb{E}^{\bar{u}}[\bar{H}_v(t)] | \mathcal{F}_t^Y], u(t) - \bar{u}(t) \rangle dt \right] \\ &= \mathbb{E} \left[\int_0^T \langle \bar{Z}(t) \bar{H}_u(t) + \mathbb{E}^{\bar{u}}[\bar{H}_v(t)], u(t) - \bar{u}(t) \rangle dt \right] \\ &= \lim_{\varepsilon \rightarrow 0} \frac{J(\bar{u}(\cdot) + \varepsilon(u(\cdot) - \bar{u}(\cdot))) - J(\bar{u}(\cdot))}{\varepsilon} \geq 0, \end{aligned}$$

which imply that (2.4) holds. The proof is complete. \square

3 Stochastic Maximum Principle in Strong Formulation

This section is devoted to establish the stochastic maximum principles of Problem 1.3. In this case, the Hamiltonian $H : [0, T] \times \Omega \times \mathbb{R}^n \times \mathbb{R}^n \times U \times U \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$ is defined by

$$\begin{aligned} H(t, x, y, u, v, p, q, \tilde{q}) &= \langle p, b(t, x, y, u, v) - \tilde{g}(t, x, y, u, v) \rangle h(t, x, y, u, v) \\ &\quad + \langle q, g(t, x, y, u, v) \rangle + \langle \tilde{q}, \tilde{g}(t, x, y, u, v) \rangle + l(t, x, y, u, v). \end{aligned} \quad (3.1)$$

Then for any admissible pair $(\bar{u}(\cdot), \bar{x}(\cdot))$, the corresponding adjoint process is defined as the solution to the following mean-field BSDE:

$$\begin{cases} d\bar{p}(t) = - \left[\bar{H}_x(t) + \mathbb{E}[\bar{H}_y(t)] \right] dt + \bar{q}(t) dW(t) + \tilde{\bar{q}}(t) dY(t), \\ \bar{P}(T) = \bar{m}_x(T) + \mathbb{E}[\bar{m}_y(T)], \end{cases} \quad (3.2)$$

where we have used the following shorthand notation

$$\begin{cases} \bar{H}(t) = H(t, \bar{x}(t), \mathbb{E}[\bar{x}(t)], \bar{u}(t), \mathbb{E}[\bar{u}(t)], \bar{p}(t), \bar{q}(t), \bar{\bar{q}}(t)), \\ \bar{m}(T) = m(\bar{x}(T), \mathbb{E}[\bar{x}(T)]). \end{cases} \quad (3.3)$$

Under Assumption 1.1 and 1.2, by Buckdahn (2009b), (3.2) admits a unique strong slution $(\bar{p}(\cdot), \bar{q}(\cdot), \bar{\bar{q}}(\cdot)) \in S_{\mathcal{F}}^2(0, T; \mathbb{R}^n) \times M_{\mathcal{F}}^2(0, T; \mathbb{R}^n) \times M_{\mathcal{F}}^2(0, T; \mathbb{R}^n)$, which is also called the adjoint process corresponding to the admissible pair $(u(\cdot), x(\cdot))$

3.1 Sufficient Conditions of Optimality

In this section, we are going to establish the sufficient Pontryagin maximum principle of Problem 1.3. To this end, we need the following Lemma.

Lemma 3.1. *Let Assumptions 1.1 and 1.2 be satisfied. Let $(u(\cdot), x(\cdot))$ and $(\bar{u}(\cdot), \bar{x}(\cdot))$ be two any given admissible pair of Problem 1.3. Let $(\bar{p}(\cdot), \bar{q}(\cdot), \bar{\bar{q}}(\cdot))$ be the adjoint process associated with the admissible pair $(\bar{u}(\cdot), \bar{x}(\cdot))$. Then for the cost functional (1.10), using the notation (3.3), we have the following presentation:*

$$\begin{aligned} & J(u(\cdot)) - J(\bar{u}(\cdot)) \\ &= \mathbb{E} \int_0^T \left[H(t) - \bar{H}(t) - \langle x^\epsilon(t) - \bar{x}(t), \bar{H}_x(t) + \mathbb{E}[\bar{H}_y(t)] \rangle \right] dt \\ & \quad + \mathbb{E} [m(T) - \bar{m}(T) - \langle x(T) - \bar{x}(T), \bar{m}_x(T) + \mathbb{E}[\bar{m}_y(T)] \rangle], \end{aligned} \quad (3.4)$$

where

$$\begin{cases} H(t) := H(t, x(t), \mathbb{E}[x(t)], u(t), \mathbb{E}[u(t)], \bar{p}(t), \bar{q}(t), \bar{\bar{q}}(t)), \\ m(T) := m(x(T), \mathbb{E}[x(T)]). \end{cases} \quad (3.5)$$

Proof. Similar to the proof of Lemma 2.3, (3.4) can be obtained by using the definition of the Hamiltonian function H (see (3.1)) and the cost functional $J(u(\cdot))$ (see (1.10)) and applying Itô formula to $\langle x(t) - \bar{x}(t), \bar{p}(t) \rangle$ and then taking expectation under the probability \mathbb{P} . Since the proof is standard, here we omit the concrete calculation. The proof is complete. \square

Next we give the sufficient condition of optimality for the existence of an optimal control of Problem 1.3.

Theorem 3.2. [Sufficient Stochastic Maximum Principle I]

Let Assumptions 1.1 and 1.2 be satisfied. Let $(\bar{u}(\cdot), \bar{x}(\cdot))$ be an admissible pair associated with the adjoint process $(\bar{p}(\cdot), \bar{q}(\cdot), \bar{\bar{q}}(\cdot))$. Suppose that

1. $H(t, x, y, u, v, \bar{p}(t), \bar{q}(t), \bar{\bar{q}}(t))$ is convex in (x, y, u, v) ,
2. $m(x, y)$ is convex in (x, y) ,
3. For any $u(\cdot) \in U_{ad}^S$,

$$\mathbb{E} \left[\langle u(t) - \bar{u}(t), \bar{H}_u(t) + \mathbb{E}[\bar{H}_v(t)] \rangle \right] \geq 0. \quad (3.6)$$

Then $(\bar{u}(\cdot), \bar{x}(\cdot))$ is an optimal pair of Problem 1.3.

Proof. Let $(u(\cdot), x(\cdot))$ be an arbitrary admissible pair. In view of Lemma 3.1, we have

$$\begin{aligned} & J(u(\cdot)) - J(\bar{u}(\cdot)) \\ &= \mathbb{E} \left[\int_0^T \left(H(t) - \bar{H}(t) - \langle x(t) - \bar{x}(t), \bar{H}_x(t) + \mathbb{E}[\bar{H}_y(t)] \rangle \right) dt \right] \\ & \quad + \mathbb{E} \left[m(T) - \bar{m}(T) - \langle x(T) - \bar{x}(T), \bar{m}_x(T) + \mathbb{E}[\bar{m}_y(T)] \rangle \right]. \end{aligned} \quad (3.7)$$

The condition 1 and 3 lead to

$$\begin{aligned} \mathbb{E}[H(t) - \bar{H}(t)] &\geq \mathbb{E} \left[\langle x(t) - \bar{x}(t), \bar{H}_x(t) + \mathbb{E}[\bar{H}_y(t)] \rangle + \langle u(t) - \bar{u}(t), \bar{H}_u(t) + \mathbb{E}[\bar{H}_v(t)] \rangle \right] \\ &\geq \mathbb{E} \left[\langle x(t) - \bar{x}(t), \bar{H}_x(t) + \mathbb{E}[\bar{H}_y(t)] \rangle \right]. \end{aligned} \quad (3.8)$$

The condition 2 arrives at

$$\mathbb{E}[m(T) - \bar{m}(T)] \geq \mathbb{E} \left[\langle x(T) - \bar{x}(T), \bar{m}_x(T) + \mathbb{E}[\bar{m}_y(T)] \rangle \right]. \quad (3.9)$$

Putting (3.8) and (3.9) into (3.7), we get

$$J(u(\cdot)) - J(\bar{u}(\cdot)) \geq 0. \quad (3.10)$$

Since $u(\cdot)$ is arbitrary, $\bar{u}(\cdot)$ is an optimal control and thus $(\bar{u}(\cdot), \bar{x}(\cdot))$ is an optimal pair. The proof is complete. \square

The convexity condition of m is sometimes too strong to hold which may limit the applicability of our sufficient maximum principle. To overcome this limitation, we note that the proof of Theorem 3.2 still holds as long as the terminal cost m is convex in an expected sense. Therefore, weakening the convexity of the m , we provide the following corollary of Theorem 3.2 as the second sufficient maximum principle.

Corollary 3.3. [*Sufficient Stochastic Maximum principle II*] *Let Assumption 1.1 and 1.2 be satisfied. Let $(\bar{u}(\cdot), \bar{x}(\cdot))$ be an admissible pair associated with the adjoint process $(\bar{p}(\cdot), \bar{q}(\cdot), \bar{\tilde{q}}(\cdot))$. Suppose that*

1. $H(t, x, y, u, v, \bar{p}(t), \bar{q}(t), \bar{\tilde{q}}(t))$ is convex in (x, y, u, v) ,
2. For any random variables $X_1, X_2, \in L^2(\Omega, \mathcal{F}, P; \mathbb{R}^n)$,

$$\mathbb{E} \left[m(X_1, \mathbb{E}[X_1]) - m(X_2, \mathbb{E}[X_2]) \right] \geq \mathbb{E} \left[\langle X_1 - X_2, m_x(X_2, \mathbb{E}[X_2]) + \mathbb{E}[m_y(X_2, \mathbb{E}[X_2])] \rangle \right],$$

3. For any $u(\cdot) \in U_{ad}^S$,

$$\mathbb{E} \left[\langle u(t) - \bar{u}(t), \bar{H}_u(t) + \mathbb{E}[\bar{H}_v(t)] \rangle \right] \geq 0, \quad (3.11)$$

then $\bar{u}(\cdot)$ is an optimal control and $\bar{x}(\cdot)$ is the corresponding optimal state.

Proof. Let $(u(\cdot), x(\cdot))$ be an arbitrary admissible pair. From the condition 2, we see that (3.9) holds. Moreover, following the same argument as the proof of Theorem 3.2, (3.7) and (3.8) also hold. Therefore, Putting (3.8) and (3.9) into (3.7), we get

$$J(u(\cdot)) - J(\bar{u}(\cdot)) \geq 0, \quad (3.12)$$

which implies that $\bar{u}(\cdot)$ is an optimal control and $\bar{x}(\cdot)$ is the corresponding optimal state. The proof is complete. \square

3.2 Necessary Conditions of Optimality

In this section we are going to represent the necessary Pontryagin maximum principle of Problem 1.3. To this end, we need the following variation formula.

Theorem 3.4. *Let Assumption 1.1 and 1.2 be satisfied. Let $(\bar{u}(\cdot), \bar{x}(\cdot))$ be an admissible pair associated with the adjoint process $(\bar{p}(\cdot), \bar{q}(\cdot), \bar{\tilde{q}}(\cdot))$. Then*

$$\begin{aligned} & \frac{d}{d\epsilon} J(\bar{u}(\cdot) + \epsilon(u(\cdot) - \bar{u}(\cdot)))|_{\epsilon=0} \\ &:= \lim_{\epsilon \rightarrow 0^+} \frac{J(\bar{u}(\cdot) + \epsilon(u(\cdot) - \bar{u}(\cdot))) - J(\bar{u}(\cdot))}{\epsilon} \\ &= \mathbb{E} \left[\int_0^T \langle \mathbb{E}[\bar{H}_u(t) + \mathbb{E}[\bar{H}_v(t)], u(t) - \bar{u}(t)] \rangle dt \right]. \end{aligned} \quad (3.13)$$

where $\epsilon \in (0, 1)$ and $u(\cdot)$ is any given admissible control.

Proof. Following an argument similar to the proof of Theorem 2.4, (3.13) can be obtained by Lemma 3.1. Here we do not repeat it. The proof is complete. \square

Then by Theorem 3.4, we get the following the necessary Pontryagin maximum principle of Problem 1.3.

Theorem 3.5. *Let Assumption 1.1 and 1.2 be satisfied. Let $(\bar{u}(\cdot); \bar{x}(\cdot))$ be an optimal pair of Problem 1.3 associated with the adjoint process $(\bar{p}(\cdot), \bar{q}(\cdot), \bar{\tilde{q}}(\cdot))$. . Then the optimality condition*

$$\left\langle \mathbb{E}[\bar{H}_u(t) + \mathbb{E}[\bar{H}_v(t)] | \mathcal{F}_t^Y], u - \bar{u}(t) \right\rangle \geq 0 \quad (3.14)$$

holds for all $u \in U$ and a.e. $(t, \omega) \in [0, T] \times \Omega$.

Proof. Since all admissible controls are $\{\mathcal{F}_t^Y\}_{t \in \mathcal{T}}$ -adapted processes, from the property of conditional expectation, Theorem 3.4 and the optimality of $\bar{u}(\cdot)$, we deduce that

$$\begin{aligned} & \mathbb{E} \left[\int_0^T \langle \mathbb{E}[\bar{H}_u(t) + \mathbb{E}[\bar{H}_v(t)] | \mathcal{F}_t^Y], u(t) - \bar{u}(t) \rangle dt \right] \\ &= \mathbb{E} \left[\int_0^T \langle \bar{H}_u(t) + \mathbb{E}[\bar{H}_v(t)], u(t) - \bar{u}(t) \rangle dt \right] \\ &= \lim_{\epsilon \rightarrow 0} \frac{J(\bar{u}(\cdot) + \epsilon(u(\cdot) - \bar{u}(\cdot))) - J(\bar{u}(\cdot))}{\epsilon} \geq 0, \end{aligned}$$

which imply that (3.14) holds. The proof is complete. \square

4 Application

In this section, we apply our stochastic maximum principle to solve a partial observed stochastic linear quadratic (LQ) optimal control problem. Let us make it more precise below. In this case, we assume the state system is the following linear mean-field SDE

$$\begin{cases} dX(t) = (A_1(t)X(t) + A_2(t)\mathbb{E}[X(t)] + B_1(t)u(t) + B_2(t)\mathbb{E}[u(t)])dt \\ \quad + (C_1(t)X(t) + C_2(t)\mathbb{E}[X(t)] + D_1(t)u(t) + D_2(t)\mathbb{E}[u(t)])dW(t) \\ \quad + (F_1(t)X(t) + F_2(t)\mathbb{E}[X(t)] + G_1(t)u(t) + G_2(t)\mathbb{E}[u(t)])dW^u(t), \\ x(0) = x \in \mathbb{R}^n, \end{cases} \quad (4.1)$$

with an observation

$$\begin{cases} dY(t) = h(t)dt + dW^u(t), \\ Y(0) = 0, \end{cases} \quad (4.2)$$

and the cost functional has the following quadratic form:

$$\begin{aligned} J(u(\cdot)) = & \mathbb{E}[\langle M_1 X(T), X(T) \rangle] + \mathbb{E}[\langle M_2 \mathbb{E}[X(T)], \mathbb{E}[X(T)] \rangle] \\ & + \mathbb{E} \left[\int_0^T \langle Q_1(s)X(s), X(s) \rangle ds \right] + \mathbb{E} \left[\int_0^T \langle Q_2(s)\mathbb{E}[X(s)], \mathbb{E}[X(s)] \rangle ds \right] \\ & + \mathbb{E} \left[\int_0^T \langle N_1(s)u(s), u(s) \rangle ds \right] + \mathbb{E} \left[\int_0^T \langle N_2(s)\mathbb{E}[u(s)], \mathbb{E}[u(s)] \rangle ds \right]. \end{aligned} \quad (4.3)$$

In this case, our control process $u(\cdot)$ is said to be an admissible stochastic process if $u(\cdot) \in M_{\mathcal{F}^Y}^2(0, T; \mathbb{R}^k)$. The set of all admissible controls is also denoted by U_{ad}^S . Note that there is no constraint on our control process, since it takes value in \mathbb{R}^k . Now we make the basic assumptions on the coefficients.

Assumption 4.1. The matrix-valued functions $A_1, A_2, C_1, C_2, F_1, F_2, Q_1, Q_2 : [0, T] \rightarrow \mathbb{R}^{n \times n}$; $B_1, B_2, D_1, D_2, G_1, G_2, : [0, T] \rightarrow \mathbb{R}^{n \times k}$; $N_1, N_2 : [0, T] \rightarrow \mathbb{R}^{k \times k}$; $h : [0, T] \rightarrow \mathbb{R}$ are uniformly bounded measurable functions. M_1 and M_2 are matrices in $\mathbb{R}^{n \times n}$.

Assumption 4.2. The matrix-valued functions $Q_1, Q_1 + Q_2, N_1, N_1 + N_2$ are a.e. nonnegative matrices, and $M_1, M_1 + M_2$ are nonnegative matrices. Moreover, $N_1, N_1 + N_2$ uniformly positive, i.e. for $\forall u \in \mathbb{R}^m$ and a.s. $t \in [0, T]$, $\langle N_1(t)u, u \rangle \geq \delta \langle u, u \rangle$ and $\langle (N_1(t) + N_2(t))u, u \rangle \geq \delta \langle u, u \rangle$, for some positive constant δ .

Then our partial observed mean-field LQ problem can be stated as follows.

Problem 4.1. Find an admissible control $\bar{u}(\cdot)$ such that

$$J(\bar{u}(\cdot)) = \inf_{u(\cdot) \in U_{ad}^S} J(u(\cdot)), \quad (4.4)$$

subject to (4.1), (4.2) and (4.3).

It is easy to check that under Assumptions 4.1 and 4.2, if we set

$$\begin{aligned}
b(t, x, y, u, v) &= A_1(t)x + A_2(t)y + B_1(t)u + B_2(t)v, \\
g(t, x, y, u, v) &= C_1(t)x + C_2(t)y + D_1(t)u + D_2(t)v, \\
\tilde{g}(t, x, y, u, v) &= F_1(t)x + F_2(t)y + G_1(t)u + G_2(t)v, \\
h(t, x, y, u, v) &= h(t), m(x, y) = (M_1x, x) + (M_2y, y) \\
l(t, x, y, u, v) &= (Q_1x, x) + (Q_2y, y) + (N_1u, u) + (N_2v, v).
\end{aligned} \tag{4.5}$$

Problem 4.1 can be regarded as a special case of Problem 1.3 and Assumptions 1.1 and 1.2 for (4.5) hold. Thus Theorem 3.2 and 3.5 can be applied to solve Problem 4.1. In this case, the Hamiltonian becomes

$$\begin{aligned}
&H(t, x, y, u, v, p, q, \tilde{q}) \\
&= \langle p, A_1(t)x + A_2(t)y + B_1(t)u + B_2(t)v - h(t)(F_1(t)x + F_2(t)y + G_1(t)u + G_2(t)v) \rangle \\
&\quad + \langle q, C_1(t)x + C_2(t)y + D_1(t)u + D_2(t)v \rangle + \langle \tilde{q}, F_1(t)x + F_2(t)y + G_1(t)u + G_2(t)v \rangle \\
&\quad + \langle Q_1x, x \rangle + \langle Q_2y, y \rangle + \langle N_1u, u \rangle + \langle N_2v, v \rangle.
\end{aligned} \tag{4.6}$$

For any admissible pair $(u(\cdot), x(\cdot))$, the corresponding adjoint equation becomes

$$\begin{cases} dp(t) = - \left[(A_1^\top(t) - h(t)F_1^\top(t))p(t) + (A_2^\top(t) - h(t)F_2^\top(t))\mathbb{E}[p(t)] + C_1^\top(t)q(t) \right. \\ \quad \left. + C_2^\top(t)\mathbb{E}[q(t)] + F_1^\top(t)\tilde{q}(t) + F_2^\top(t)\mathbb{E}[\tilde{q}(t)] + 2Q_1(t)X(t) + 2Q_2(t)\mathbb{E}[X(t)] \right] dt \\ \quad + q(t)dW(t) + \tilde{q}(t)dY(t), \\ p(T) = 2M_1X(T) + 2M_2\mathbb{E}[X(T)]. \end{cases} \tag{4.7}$$

The following result gives the existence and uniqueness of the optimal control of Problem 4.1.

Theorem 4.2. *Let Assumptions 4.1 and 4.2 be satisfied. Then Problem 4.1 has a unique optimal control.*

Proof. Since the admissible control set $U_{ad}^S = M_{\mathcal{F}_Y}^2(0, T; \mathbb{R}^k)$ is a Hilbert space, thus a reflexive Banach space, to prove the existence and uniqueness of the optimal control, by the classic optimality principle (see Proposition 2.12 of Ekeland and Témam (1976)), it needs only to prove that over U_{ad}^S , the cost functional $J(u(\cdot))$ is the strictly convex, coercive and lower-semi continuous. Indeed, by the a priori estimate (1.13) and (1.15), over U_{ad}^S , we can show that the cost functional $J(u(\cdot))$ is continuous and hence lower-semi continuous. On the other hand, since the weighting matrices in the cost functional are not random, from the definition of $J(u(\cdot))$ (see (4.3)) and by a simple calculation, we can get that

$$\begin{aligned}
J(u(\cdot)) &= \mathbb{E} \left[\int_0^T \left(\langle Q_1(t)(X(t) - \mathbb{E}[X(t)]), X(t) - \mathbb{E}[X(t)] \rangle + \langle (Q_1 + Q_2)(t)\mathbb{E}[X(t)], \mathbb{E}[X(t)] \rangle \right. \right. \\
&\quad \left. \left. + \langle N_1(t)(u(t) - \mathbb{E}[u(t)]), u(t) - \mathbb{E}[u(t)] \rangle + \langle (N_1(t) + N_2(t))\mathbb{E}[u(t)], \mathbb{E}[u(t)] \rangle \right) dt \right] \\
&\quad + \mathbb{E} \left[\langle M_1(X(T) - \mathbb{E}[X(T)]), X(T) - \mathbb{E}[X(T)] \rangle + \langle (M_1 + M_2)\mathbb{E}[X(T)], \mathbb{E}[X(T)] \rangle \right].
\end{aligned} \tag{4.8}$$

Thus the cost functional $J(u(\cdot))$ over U_{ad}^S is convex from the nonnegativity of the $N_1, N_1 + N_2, Q_1, Q_1 + Q_2, M_1, M_1 + M_2$. Actually, since N_1 and $N_1 + N_2$ are uniformly positive, $J(u(\cdot))$ is strictly convex. Furthermore, it follows from the nonnegativity of $M_1, M_1 + M_2$ and $Q_1, Q_1 + Q_2$ and the uniformly strictly positivity of $N_1, N_1 + N_2$, that

$$\begin{aligned}
J(u(\cdot)) &\geq \mathbb{E} \left[\int_0^T \left(\langle N(t)(u(t) - \mathbb{E}[u(t)]), u(t) - \mathbb{E}[u(t)] \rangle + \langle (N(t) + \bar{N}(t))\mathbb{E}[u(t)], \mathbb{E}[u(t)] \rangle \right) dt \right] \\
&\geq \delta \mathbb{E} \left[\int_0^T \langle u(t) - \mathbb{E}[u(t)], u(t) - \mathbb{E}[u(t)] \rangle dt \right] + \delta \mathbb{E} \left[\int_0^T \langle \mathbb{E}[u(t)], \mathbb{E}[u(t)] \rangle dt \right] \\
&= \delta \mathbb{E} \left[\int_0^T |u(t)|^2 dt \right] \\
&= \delta \|u(\cdot)\|_{U_{ad}^S}^2,
\end{aligned} \tag{4.9}$$

which implies that $J(u(\cdot))$ is coercive, i.e.,

$$\lim_{\|u(\cdot)\|_{U_{ad}^S} \rightarrow \infty} J(u(\cdot)) = \infty.$$

In summary, the cost functional $J(u(\cdot))$ is strictly convex, coercive, lower-semi continuous over the reflexive Banach space U_{ad}^S . The proof is complete. \square

In the following, applying the maximum principle to our LQ problem, we give the dual presentation of the optimal control in terms of the corresponding adjoint process.

Theorem 4.3. *Let Assumptions 4.1 and 4.2 be satisfied. Then, a necessary and sufficient condition for an admissible pair $(u(\cdot); x(\cdot))$ to be an optimal pair of Problem 4.1 is that the admissible control $u(\cdot)$ satisfies*

$$\begin{aligned}
&2N_1(t)u(t) + 2N_2(t)\mathbb{E}[u(t)] + (B_1^\top(t) - h(t)G_1^\top(t))\mathbb{E}[p(t)|\mathcal{F}_t^Y] + (B_2^\top(t) - h(t)G_2^\top(t))\mathbb{E}[p(t)] \\
&+ D_1^\top(t)\mathbb{E}[q(t)|\mathcal{F}_t^Y] + D_2^\top(t)\mathbb{E}[q(t)] = 0, \quad a.e.a.s.,
\end{aligned} \tag{4.10}$$

where $(p(\cdot), q(\cdot), \tilde{q}(\cdot))$ is the solution to the adjoint equation (4.7) corresponding to $(u(\cdot), X(\cdot))$.

Proof. For the necessary part, let $(u(\cdot), x(\cdot))$ be an optimal pair associated with the adjoint process $(p(\cdot), q(\cdot), \tilde{q}(\cdot))$. Since there is no constraints on the control processes, then from the necessary optimality condition (3.14) (see Theorem 3.5), we get that

$$\begin{aligned}
&\mathbb{E} \left[H_u(t, x(t), \mathbb{E}[x(t)], u(t), \mathbb{E}[u(t)], p(t), q(t), \tilde{q}(t)) | \mathcal{F}_t^Y \right] \\
&+ \mathbb{E} \left[H_v(t, x(t), \mathbb{E}[x(t)], u(t), \mathbb{E}[u(t)], p(t), q(t), \tilde{q}(t)) \right] = 0,
\end{aligned} \tag{4.11}$$

which leads to (4.10) (recalling the definition (4.6) of Hamiltonian H).

For the sufficient part, let $(u(\cdot), X(\cdot))$ be an admissible pair associated with the adjoint process $(p(\cdot), q(\cdot), \tilde{q}(\cdot))$ and assume the condition (4.10) holds. From the definition of H (see (4.6)), the condition (4.10) implies (4.11) holds. Thus, since any admissible control is \mathcal{F}_t^Y -adapted

process. by(4.11) , for any other admissible control $v(\cdot)$, from the property of conditional expectation, we have

$$\begin{aligned}
& \mathbb{E} \left[\left\langle v(t) - u(t), H_u(t, x(t), \mathbb{E}[x(t)], u(t), \mathbb{E}[u(t)], p(t), q(t), \tilde{q}(t)) \right. \right. \\
& \quad \left. \left. + \mathbb{E} \left[H_v(t, x(t), \mathbb{E}[x(t)], u(t), \mathbb{E}[u(t)], p(t), q(t), \tilde{q}(t)) \right] \right\rangle \right] \\
&= \mathbb{E} \left[\left\langle v(t) - \bar{u}(t), \mathbb{E} \left[H_u(t, x(t), \mathbb{E}[x(t)], u(t), \mathbb{E}[u(t)], p(t), q(t), \tilde{q}(t)) | \mathcal{F}_t^Y \right] \right. \right. \\
& \quad \left. \left. + \mathbb{E} \left[H_v(t, x(t), \mathbb{E}[x(t)], u(t), \mathbb{E}[u(t)], p(t), q(t), \tilde{q}(t)) \right] \right\rangle \right] \\
&= 0,
\end{aligned} \tag{4.12}$$

which implies that the condition 3 in Theorem 3.2 holds. Moreover, under Assumptions 4.1 and 4.2, it is easy to check that all other conditions in Theorem 3.2 are satisfied. Therefore, by Theorem 3.2, we conclude that $(u(\cdot), x(\cdot))$ is an optimal control pair. The proof is complete. \square

From the above, we end up the following optimality system

$$\left\{ \begin{array}{l} dX(t) = (A_1(t)X(t) + A_2(t)\mathbb{E}[X(t)] + B_1(t)u(t) + B_2(t)\mathbb{E}[u(t)])dt \\ \quad + (C_1(t)X(t) + \bar{C}_2(t)\mathbb{E}[X(t)] + D_1(t)u(t) + D_2(t)\mathbb{E}[u(t)])dW(t) \\ \quad + (F_1(t)X(t) + F_2(t)\mathbb{E}[X(t)] + G_1(t)u(t) + G_2(t)\mathbb{E}[u(t)])dW^u(t), \\ dY(t) = h(t)dt + dW^u(t), \\ dp(t) = - \left[(A_1^\top(t) - h(t)F_1^\top(t))p(t) + (A_2^\top(t) - h(t)F_2^\top(t))\mathbb{E}[p(t)] + C_1^\top(t)q(t) \right. \\ \quad \left. + C_2^\top(t)\mathbb{E}[q(t)] + F_1^\top(t)\tilde{q}(t) + F_2^\top(t)\mathbb{E}[\tilde{q}(t)] + 2Q_1(t)X(t) + 2Q_2(t)\mathbb{E}[X(t)] \right] dt \\ \quad + q(t)dW(t) + \tilde{q}(t)dY(t), \\ x(0) = x, p(T) = 2M_1X(T) + 2M_2\mathbb{E}[X(T)], Y(0) = 0, \\ 2N_1(t)u(t) + 2N_2(t)\mathbb{E}[u(t)] + (B_1^\top(t) - h(t)G_1(t))\mathbb{E}[p(t)|\mathcal{F}_t^Y] + (B_2^\top(t) - h(t)G_2(t))\mathbb{E}[p(t)] \\ \quad + D_1^\top(t)\mathbb{E}[q(t)|\mathcal{F}_t^Y] + D_2^\top(t)\mathbb{E}[q(t)] = 0. \end{array} \right. \tag{4.13}$$

This is a fully coupled forward-backward stochastic differential equations of mean-field type. Note that the coupling comes from the last relation (which is essentially the maximum condition in the Pontryagin type maximum principle). The 5-tuple $(u(\cdot), x(\cdot), p(\cdot), q(\cdot), \tilde{q}(\cdot))$ of \mathcal{F}_t -adapted processes satisfying the above is called an adapted solution of (4.13). Then by Theorem 4.3, we can directly obtain the following equivalence between the solvability of optimality system (4.13) and the existence and unique of the optimal control of Problem 4.1.

Corollary 4.4. *Let Assumptions 4.1 and 4.2 be satisfied. Then, a necessary and sufficient condition for that the optimality system (4.13) has a unique solution strong solution $(u(\cdot), x(\cdot), p(\cdot), q(\cdot), \tilde{q}(\cdot)) \in M_{\mathcal{F}^u}^2(0, T; \mathbb{R}^k) \times S_{\mathcal{F}}^2(0, T; \mathbb{R}^n) \times S_{\mathcal{F}}^2(0, T; \mathbb{R}^n) \times M_{\mathcal{F}}^2(0, T; \mathbb{R}^n) \times M_{\mathcal{F}}^2(0, T; \mathbb{R}^n)$ is that $(u(\cdot); x(\cdot))$ is a unique optimal pair of Problem 4.1.*

Remark 4.1. In summary, the optimality system (4.13) completely characterizes the optimal control of Problem 4.1. Therefore, solving Problem 4.1 is equivalent to solving the optimality

system, moreover, the unique optimal control can be given by (4.10). Taking expectation on (4.10), we have

$$\begin{aligned} & 2(N_1(t) + N_2(t))\mathbb{E}[u(t)] + (B_1^\top(t) + B_2^\top(t) - h(t)G_1^\top(t) - h(t)G_2^\top(t))\mathbb{E}[p(t)] \\ & + (D_1^\top(t) + D_2^\top(t))\mathbb{E}[q(t)] = 0, \quad a.e.a.s., \end{aligned} \quad (4.14)$$

which implies

$$\begin{aligned} \mathbb{E}[u(t)] = & -\frac{1}{2}(N_1(t) + N_2(t))^{-1} \left[(B_1(t) + B_2(t) - h(t)G_1(t) - h(t)G_2(t))^\top \mathbb{E}[p(t)] \right. \\ & \left. + (D_1(t) + D_2(t))^\top \mathbb{E}[q(t)] \right], a.s. \end{aligned} \quad (4.15)$$

Putting (4.15) into (4.10), we get that

$$\begin{aligned} 2N_1(t)u(t) = & -2N_2(t)\mathbb{E}[u(t)] - (B_1^\top(t) - h(t)G_1^\top(t))\mathbb{E}[p(t)|\mathcal{F}_t^Y] - (B_2^\top(t) - h(t)G_2^\top(t))\mathbb{E}[p(t)] \\ & - D_1^\top(t)\mathbb{E}[q(t)|\mathcal{F}_t^Y] - D_2^\top(t)\mathbb{E}[q(t)], \end{aligned} \quad (4.16)$$

which imply that the optimal control $u(\cdot)$ has the following explicit dual presentation

$$\begin{aligned} \bar{u}(t) = & -\frac{1}{2}N_1^{-1}(t) \left\{ (B_1^\top(t) - h(t)G_1^\top(t))\mathbb{E}[p(t)|\mathcal{F}_t^Y] + (B_2^\top(t) - h(t)G_2^\top(t))\mathbb{E}[p(t)] \right. \\ & + D_1^\top(t)\mathbb{E}[q(t)|\mathcal{F}_t^Y] + D_2^\top(t)\mathbb{E}[q(t)] \\ & + N_2(t)(N_1(t) + N_2(t))^{-1} \left[(B_1(t) + B_2(t) - h(t)G_1(t) - h(t)G_2(t))^\top \mathbb{E}[p(t)] \right. \\ & \left. \left. + (D_1(t) + D_2(t))^\top \mathbb{E}[q(t)] \right] \right\}, \quad a.e.a.s. \end{aligned} \quad (4.17)$$

In the following, we will give the state feedback representation of the optimal control.

Theorem 4.5. *Let Assumptions 4.1 and 4.2 be satisfied. Let $(\bar{u}(\cdot), \bar{x}(\cdot))$ be the optimal pair. Then the optimal control $\bar{u}(\cdot)$ has the following state feedback representation:*

$$\begin{aligned} & \bar{u}(t) \\ = & -\Sigma_0^{-1}(t) \left[(B_1^\top(t) - h(t)G_1^\top(t))P(t) + D_1^\top(t)P(t)C_1(t) \right] \left[\mathbb{E}[\bar{x}(t)|\mathcal{F}_t^Y] - \mathbb{E}[\bar{x}(t)] \right] \\ & - \Sigma_2^{-1}(t) \left[(B_1^\top(t) + B_2^\top(t) - h(t)(G_1^\top(t) + G_2^\top(t)))\Pi(t) \right. \\ & \left. + (D_1^\top(t) + D_2^\top(t))P(t)(C_1(t) + C_2(t)) \right] \mathbb{E}[\bar{x}(t)], \end{aligned} \quad (4.18)$$

where

$$\begin{aligned} \Sigma_0(t) = & 2N_1(t) + D_1^\top(t)PD_1(t), \\ \Sigma_2(t) = & 2(N_1(t) + N_2(t)) + (D_1^\top(t) + D_2^\top(t))P(t)(D_1(t) + D_2(t)), \end{aligned} \quad (4.19)$$

$P(\cdot)$ and $\Pi(\cdot)$ are the solutions to the following Riccati equations, respectively:

$$\left\{ \begin{array}{l} (\dot{P}(t) + P(t)A_1(t) + A_1^\top(t)P(t) + C_1^\top(t)P(t)C(t) + 2Q_1(t) \\ - \left[P(t)(B_1(t) - h(t)G_1(t)) + C_1^\top(t)P(t)D_1(t) \right] \Sigma_0^{-1}(t) \\ \cdot \left[(B_1^\top(t) - h(t)G_1^\top(t))P(t) + D_1^\top P(t)C_1(t) \right] = 0, \\ P(T) = M_1 \end{array} \right. \quad (4.20)$$

and

$$\left\{ \begin{array}{l} \dot{\Pi}(t) + \Pi(t)(A_1(t) + A_2(t)) + (A_1^\top(t) + A_2^\top(t))\Pi(t) + (C_1^\top(t) + C_2^\top(t))P(t)(C_1(t) + C_2(t)) \\ + 2(Q_1 + Q_2) \\ - \left[\Pi(t)(B_1(t) + B_2(t) - h(t)(G_1(t) + G_2(t))) + (C_1^\top(t) + C_2^\top(t))P(t)(D_1(t) + D_2(t)) \right] \\ \cdot \Sigma_2^{-1}(t) \cdot \left[(B_1^\top(t) + B_2^\top(t) - h(t)(G_1^\top(t) + G_2^\top(t)))\Pi(t) + (D_1^\top(t) + D_2^\top(t))P(C_1(t) + C_2(t)) \right] \\ = 0, \\ \Pi(T) = M_1 + M_2. \end{array} \right. \quad (4.21)$$

Moreover,

$$\inf_{u(\cdot) \in U_{ad}^S} J(u(\cdot)) = \langle \Pi(0)x, x \rangle. \quad (4.22)$$

Proof. To unburden our notation, define

$$\hat{B}_1 = B_1(t) - h(t)G_1(t), \hat{B}_2 = B_2(t) - h(t)G_2(t). \quad (4.23)$$

The proof can be obtained by the classic technique of completing squares. Indeed, let $(u(\cdot), x(\cdot))$ be any given admissible pair. From Yong (2013), we know that the Riccati equations (4.20) and (4.21) have a unique solution $P(\cdot)$ and $\Pi(\cdot)$, respectively. Then following the same argument as that of Theorem 4.2 of Yong (2013), we get that (suppressing t)

$$\begin{aligned} & J(u(\cdot)) - \langle \Pi(0)x, x \rangle \\ &= \mathbb{E} \int_0^T \left\{ \left| \Sigma_0^{\frac{1}{2}} \left[u - \mathbb{E}[u] + \Sigma_0^{-1}(\hat{B}_1^\top P + D_1^\top P C_1)(X - \mathbb{E}[X]) \right] \right|^2 \right\} dt \\ &+ \mathbb{E} \int_0^T \left\{ \left| \Sigma_2^{\frac{1}{2}} \left[\mathbb{E}[u] + \Sigma_2^{-1} \left((\hat{B}_1^\top + \hat{B}_2^\top)P + (D_1^\top + D_2^\top)P(C_1 + C_2) \right) \mathbb{E}[X] \right] \right|^2 \right\} dt. \end{aligned} \quad (4.24)$$

Thus by the well-known KallianpurStriebel formula in Kallianpur (2013), we know that the minimum $J(u(\cdot))$ over all \mathcal{F}_t^Y -measurable process $u(t)$ is attained at

$$u(t) - \mathbb{E}[u(t)] = -\Sigma_0^{-1}(\hat{B}_1^\top P + D_1^\top P C_1)(\mathbb{E}[X(t)|\mathcal{F}_t^Y] - \mathbb{E}[X]) \quad (4.25)$$

and

$$\mathbb{E}[u] = -\Sigma_2^{-1}(\hat{B}_1^\top + \hat{B}_2^\top)P + (D_1^\top + D_2^\top)P(C_1 + C_2)\mathbb{E}[u(t)], \quad (4.26)$$

and the minimum value is $\langle \Pi(0)x, x \rangle$. Therefore, combining (4.25) and (4.26), we get that the optimal control $\bar{u}(\cdot)$ has the state feedback representation (4.18) The proof is complete. \square

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